Variational Iteration Method For Solving Nonlinear Fractional Integro-Differential Equations

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***Abstract*:** In this study, a modification of variational iteration method is applied to solve nonlinear fractional integro-differential equations. The fractional derivative is considered in the Caputo sense. The approximate solutions are calculated in the form of a convergent series with easily computable components. Through examples, we will see the modified method performs extremely effective in terms of efficiency and simplicity to solve nonlinear fractional integro-differential equations.

***Keywords*:** Nonlinear fractional integro-differential equations; Fractional derivative; Variational iteration method.

1. Introduction

In recent years, it has turned out that many phenomena in physics, engineering, chemistry, and other sciences can be described very successfully by models using mathematical tool from fractional calculus, such as, frequency dependent damping behavior of materials, diffusion processes, motion of a large thin plate in a Newtonian fluid creeping and relaxation functions for viscoelastic materials. etc. In addition to use of fractional differentiation for the mathematical modeling of real world physical problems has been widespread in recent years, e.g. the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, measurement of viscoelastic material properties, etc.

Most fractional differential equations do not have exact analytic solutions. There are only a few techniques for the solution of fractional integro-differential equations. Three of them are the Adomian decomposition method [1], the collocation method [2], and the fractional differential transform method [3]. The variational iteration method was proposed by he [4-10] and has found a wide application for the solution of linear and nonlinear differential equations, for example, linear fractional integro-differential equations[4], nonlinear wave equations [5], Fokker–Planck equation [6], Helmholtz equation [7], klein-Gordon equations [8], integro-differential equations [9], and space and time-fractional KdV equation [10].

In the study presented, fractional differentiation and integration are understood in Caputo sense because of its applicability to real world physical problems. We will set a new modified variational iteration method to solve nonlinear fractional integro-differential equations. It will show the modification of the method is a useful and simplify tool to solve nonlinear fractional integro-differential equations as used in other fields.

2. Basic Definitions

In this section, let us recall essentials of fractional calculus first. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. We have well known definitions of a fractional derivative of order $\alpha > 0$ such as Riemann–Liouville, Grunwald–Letnikow, Caputo and Generalized Functions Approach [11,12]. The most commonly used definitions are the Riemann–Liouville and Caputo. For the purpose of this paper the Caputo's definition of fractional differentiation will be used, taking the advantage of Caputo's approach that the initial conditions for fractional differential equations with Caputo's derivatives take on the traditional form as for integer-order differential equations. We give some basic definitions and properties of the fractional calculus theory which were used through paper.

Definition 2.1. A real function $f(x), x > 0,$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $(p > \mu)$, such that $f(x) = x^p f_1(x),$ where $f_1(x) \in C[0,\infty),$ and it said to be in the space C_μ^m iff $f^m \in C_\mu, m \in N.$

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \ge 0,$ of a function $f \in C_\mu, \mu \ge -1,$ is defined as

$$J_0^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x \left(x-t\right)^{\nu-1} f(t)dt, \quad \nu > 0,$$

$J^0 f(x) = f(x).$

It has the following properties:

For $f \in C_\mu, \mu \ge -1, \alpha, \beta \ge 0$ and $\gamma > 1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$
3. $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$

The Riemann–Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for the physical problems of the real world since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^v f(x) = J_a^{m-v} D^m f(x) = \frac{1}{\Gamma(m-v)} \int_0^x (x-t)^{m-v-1} f^{(m)}(t) dt,$$

for $m-1 < v < m,\ m \in N,\ x > 0,\ f \in C_{-1}^m.$

Lemma2.1. If $m-1 < \alpha < m,\ m \in N$ and $f \in C_\mu^m, \mu \ge -1,$ then

$$D_*^\alpha J^\alpha f(x) = f(x),$$

$$J^\alpha D_*^v f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+) \frac{x^k}{k!}, \ \text{x>0}.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

Definition 2.4. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_{*t}^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} =$$

$$\begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi, & \text{for } m-1 < \alpha < m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in N \end{cases}$$

and the space-fractional derivative operator of order $\beta > 0$ is defined as

$$D_{*x}^\alpha u(x,t) = \frac{\partial^\beta u(x,t)}{\partial x^\beta} =$$

$$\begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^x (x-\theta)^{m-\beta-1} \frac{\partial^m u(\theta,t)}{\partial \theta^m} d\theta, & \text{for } m-1 < \beta < m, \\ \dfrac{\partial^m u(x,t)}{\partial x^m}, & \text{for } \beta = m \in N. \end{cases}$$

3. Modification of the Variational Iteration Method

Concerning the general fractional integro-differential equation of the type

$$D^\alpha y(t) = f\left(t, y(t), \int_0^t k(s,y) ds\right) \quad (1)$$

where D^α is the derivative of $y(t)$ in the sense of Caputo, and $n-1 < \alpha < n \ (n \in N)$, subject to the initial condition

$$y(0) = c.$$

According to the variational iteration method (VIM), we can construct the following correction functional

$$y_{n+1}(t) = y_n(t) + I^\alpha F(t) \quad (2)$$

where $F(t) = \lambda \left[D_*^\alpha y_n(t) - f\left(t, y_n, \int_0^t k(s, y_n) ds\right) dt \right],$ $y_n(t)$ is the n th approximation, and I^α

is Riemann-Liouville`s fractional integrate.

The lagrenge multiplier can not easy identified through (2), so approximation of the corrrection functional can be expressed as follows

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda \left\{ \frac{d^n y_n(t)}{dt^n} - f\left(t, y_n(t), \int_0^t k(s, y_n) ds\right) \right\} dt. \quad (3)$$

Then the Lagrange multiplier can be easily determined by the variational theory in (3).

λ is a general Lagrange multiplier [13]. Lagrange multipliers

$$\lambda = 1, \quad \text{for} \quad n = 1.$$

Substituting the identified Lagrange multiplier into (2) result in the following iteration procedures

$$y_{n+1}(t) = y_n(t) - I^\alpha \left\{ D^\alpha y_n(t) - f\left(t, y_n(t), \int_0^t k(s, y_n) ds\right) \right\}, (n = 0,1,2,...).$$

4. Numerical Experiment

In this section, we apply VIM to solve a nonlinear fractional integro-differential equations. All the results are calculated by using the symbolic computation software Maple.

Example

Consider the following system of nonlinear fractional integral–differential equations, with initial values

$(n_1)_0(0) = N_1, \quad (n_2)_0(0) = N_2$.

$$D_*^{\alpha} n_1(t) = n_1\left(K_1 - \gamma_1 n_2 - \int_{t-T_0}^{t} f(t-s) n_2(s) ds \right), \quad K_1 > 0, \; 0 < \alpha \le 1,$$

$$D_*^{\alpha} n_2(t) = n_2\left(-K_2 - \gamma_2 n_1 - \int_{t-T_0}^{t} f(t-s) n_1(s) ds \right), \quad K_2 > 0.$$

To solve this system by VIM, let us consider;

$$(n_1)_{n+1}(t) = (n_1)_n(t) + I^{\alpha}\left\{ \lambda\left[D_*^{\alpha}(n_1)_n(t) - g\left[(n_1)_n(t) \right] \right] \right\},$$

$$(n_2)_{n+1}(t) = (n_2)_n(t) + I^{\alpha}\left\{ \lambda\left[D_*^{\alpha}(n_2)_n(t) - g\left[(n_2)_n(t) \right] \right] \right\},$$

where

$$g\left[(n_1)_n(t) \right] = n_1\left(K_1 - \gamma_1 n_2 - \int_{t-T_0}^{t} f(t-s) n_2(s) ds \right),$$

$$g\left[(n_2)_n(t) \right] = n_2\left(K_2 - \gamma_2 n_1 - \int_{t-T_0}^{t} f(t-s) n_1(s) ds \right),$$

$(n_1)_n(t)$ and $(n_2)_n(t)$ are n th approximation.
We start with

$$(n_1)_0(0) = N_1, \; (n_2)_0(0) = N_2,$$

by the variational iteration formula, we have

$$n_1(t) = N_1 + \frac{N_1\left[K_1 - \gamma_1 N_2 - N_2(1-e^{-T_0}) \right] t^{\alpha}}{\Gamma(\alpha+1)} + \frac{N_1 K_1^2 t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$n_2(t) = N_2 + \frac{N_2\left[K_2 - \gamma_2 N_1 - N_1(1-e^{-T_0}) \right] t^{\alpha}}{\Gamma(\alpha+1)} + \frac{N_2 K_2^2 t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

When $\alpha = 1$, then we have

$$n_1(t) = N_1 + N_1\left[K_1 - \gamma_1 N_2 - N_2(1-e^{-T_0}) \right] t + 0.5 N_1 K_1^2 t^2,$$
$$n_2(t) = N_2 + N_2\left[K_2 - \gamma_2 N_1 - N_1(1-e^{-T_0}) \right] t + 0.5 N_2 K_2^2 t^2,$$

which is the same solution given by Biazar [14].

5. Conclusion

In this paper, we applied the modified variational iteration method for solving the nonlinear fractional integro-differential equations. Comparison with other traditional methods, the simplicity of the method and the obtained exact results show that the modified variational iteration method is a powerful mathematical tool for solving nonlinear fractional integro-differential equations. The method was used in a direct way without using linearization, perturbation or restrictive assumptions. It may be concluded that the method is very powerful and efficient in finding analytical as well as numericalsolutions for wide classes of nonlinear fractional integro-differential equations. It provides more realistic series solutions that converge very rapidly in real physical problems.

6. References

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